Permutations:

① A permutation of the set S is a map $\pi: S \to S$ 1-1 and onto

② S_n = permutations of $\{1, 2, \ldots, n\}$ forms a group with operation concatenation

③ Any permutation is a product of disjoint cycles

$(1\,5\,4)(2\,7)$

$$\rightsquigarrow \begin{array}{l} 1 \to 5 \\ 2 \to 7 \\ 3 \to 3 \\ 4 \to 1 \\ 5 \to 4 \\ 6 \to 6 \\ 7 \to 2 \end{array}$$

<u>lemma:</u> $\pi \in S_n$

$\alpha = (a_1\ a_2 \ldots a_r)$ a cycle

$\Rightarrow \quad \pi \alpha \pi^{-1} = (\pi(a_1)\ \pi(a_2) \ldots \pi(a_r))$

proof to show: $\pi \alpha \pi^{-1}(b) = \begin{cases} b & \text{if } b \notin \{\pi(a_1), \ldots \pi(a_r)\} \\ \pi(a_{i+1}) & \text{if } b = \pi(a_i), i<r \\ \pi(a_1) & \text{if } b = \pi(a_r) \end{cases}$

<u>case 1</u> $\quad b \notin \{\pi(a_1), \ldots \pi(a_r)\}$

$\Rightarrow \pi^{-1}(b) \notin \{a_1, \ldots a_r\}$

$\Rightarrow \alpha(\pi^{-1}(b)) = \pi^{-1}(b)$

$\Rightarrow \pi \alpha \pi^{-1}(b) = \pi \pi^{-1}(b) = b$ ✓

<u>case 2</u> $\quad b = \pi(a_i)$

$\Rightarrow \pi \alpha \pi^{-1}(b) = \pi \alpha \cancel{\pi^{-1}\pi}(a_i) = \pi \alpha(a_i) = \pi(a_{i+1})$ ✓

example: $\pi = (1\,5)(2\,3\,4)$

$\alpha = (1\;4\;2)$

$\Rightarrow \quad \pi\alpha\pi^{-1} = (\pi(1)\;\pi(4)\;\pi(2))$

$= (5\;2\;3)$

<u>Theorem</u> α, β disjoint cycles

$\Rightarrow \quad \alpha\beta = \beta\alpha$

Proof $\quad \beta = (b_1 \ldots b_s)$

α, β disjoint $\Rightarrow \quad \alpha(b_i) = b_i \qquad 1 \leq i \leq s$

$\Rightarrow \quad \alpha\beta\alpha^{-1} = (\alpha(b_1) \ldots \alpha(b_s))$

$= (b_1 \ldots b_s)$

$= \beta$

$\Rightarrow \quad \alpha\beta\alpha^{-1} = \beta \quad \Rightarrow \quad \alpha\beta = \beta\alpha$

✓

Theorem (order of a permutation)

Assume $\pi = \alpha_1 \alpha_2 . \alpha_m$, each α_i some cycle of length r_i

(length of $\alpha = (a_1 \dots a_r)$ defined to be r)

$\Rightarrow \operatorname{ord}(\pi) = \operatorname{lcm}(r_1, r_2 \dots, r_m)$

example. $\operatorname{ord}\big(\underbrace{(1\ 5\ 2)}_{\text{length}=3} \underbrace{(3\ \ 7\ 9\ 4)}_{\text{length}=4} \big) = \operatorname{lcm}(3,4) = 12$

Proof ⓐ If $\pi = \alpha = (a_1 \dots a_r) \Rightarrow \pi^j(a_1) = a_{1+j}$ if $j < r$

(check by ind. on j !)

$\Rightarrow \pi^j \neq id$ for $j < r$

$\pi^r(a_i) = a_i$

↑ check!

(b) assume $\pi = \alpha\beta$

$\alpha = (a_1 \dots a_s)$
$\beta = (b_1 \dots b_r)$ disjoint

$\Rightarrow \alpha\beta = \beta\alpha$ (previous theorem)

$\Rightarrow (\alpha\beta)^k = \alpha^k \beta^k$ for all $k \geq 1$

let $n = \mathrm{lcm}(r, s)$

- $\pi^n = (\alpha\beta)^n = \alpha^n \beta^n = \varepsilon \cdot \varepsilon = \varepsilon$

aside: $\mathrm{ord}(\alpha) = s \mid n = \mathrm{lcm}(s, r)$

$\Rightarrow \alpha^n = id.$ ($= \varepsilon$ notation in book)

same argument: $\beta^n = id = \varepsilon$

$\Rightarrow \mathrm{ord}(\pi) \mid n$

- let $1 \le k < n = \operatorname{lcm}(r, s)$

$\Rightarrow$ not divisible by both s and r

say $s \nmid k$

$\Rightarrow k = qs + r \quad 1 \le r < s$

$\Rightarrow \pi^k = \alpha^k \beta^k$

$= \alpha^r \beta^k$

$\Rightarrow \pi^k(a_1) = \alpha^r \beta^k(a_1) = \alpha^r(a_1) = a_{r+1} \ne a_1$

β disjoint with α
i.e. does not move any a_i

$\Rightarrow \pi^k \ne \varepsilon$

(c) general case: $\pi = \alpha_1 \dots \alpha_m$ m-cycles.

same strategy.

If $k < n = \operatorname{lcm}(r_1, r_2 \dots r_m)$ $\quad r_i =$ length of α_i

then $\exists i$ s.t. $r_i \nmid k$

$\Rightarrow \alpha_i^k(b) \ne b$ for any number b in cycle of α_i

Lemma Any permutation can be written as a product of 2-cycles

[example: $(123) \overset{!}{=} (12)(23)$]

Proof. **Claim:** If $\alpha = (a_1 a_2 \dots a_r)$

$\Rightarrow \quad \alpha = (a_1 a_2)(a_2 a_3) \dots (a_{r-1} a_r)$

proof by ind. on r:

we calculate $(a_1 \dots a_{r-1})(a_{r-1} a_r)$

		apply $(a_{r-1} a_r)$	apply $(a_1 \dots a_{r-1})$
	a_r	$\longrightarrow a_{r-1}$	$\longrightarrow a_1$
	a_{r-1}	$\longrightarrow a_r$	$\longrightarrow a_r$
$i < r-1$:	a_i	$\longrightarrow a_i$	$\longrightarrow a_{i+1}$

$\Rightarrow \quad (a_1 \dots a_{r-1})(a_{r-1} a_r) = (a_1 a_2 \dots a_r)$

using ind. ass for $(a_1 \dots a_{r-1}) = (a_1 a_2)(a_2 a_3) \dots (a_{r-2} a_{r-1})$

$\Rightarrow$ claim

for general permutation $\pi = \alpha_1 \alpha_2 \dots \alpha_r$
apply this to each cycle! ✓

Last lemma $\Rightarrow$ π can be written as product of 2-cycles

Lots of different ways how to do that!

$$(123) = (12)(23)$$
$$= (23)(13)$$
$$= (23)(13)(12)(12)$$

<u>Theorem</u> Assume $\pi = \beta_1 \beta_2 \dots \beta_r$ $\quad$ β_i, α_j all 2-cycles
$$= \alpha_1 \alpha_2 \dots \alpha_s$$

$\Rightarrow$ either both r and s are odd
or both r and s are even